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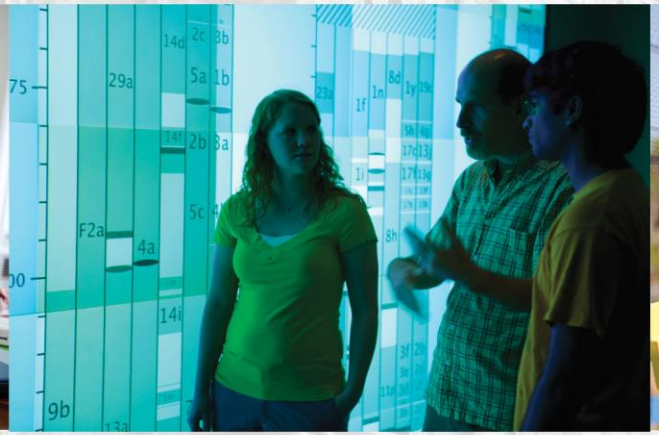
Frequency Domain Analysis of Linear Circuits Using Synchronous Detection

Physics 401, Fall 2016.

Eugene V. Colla

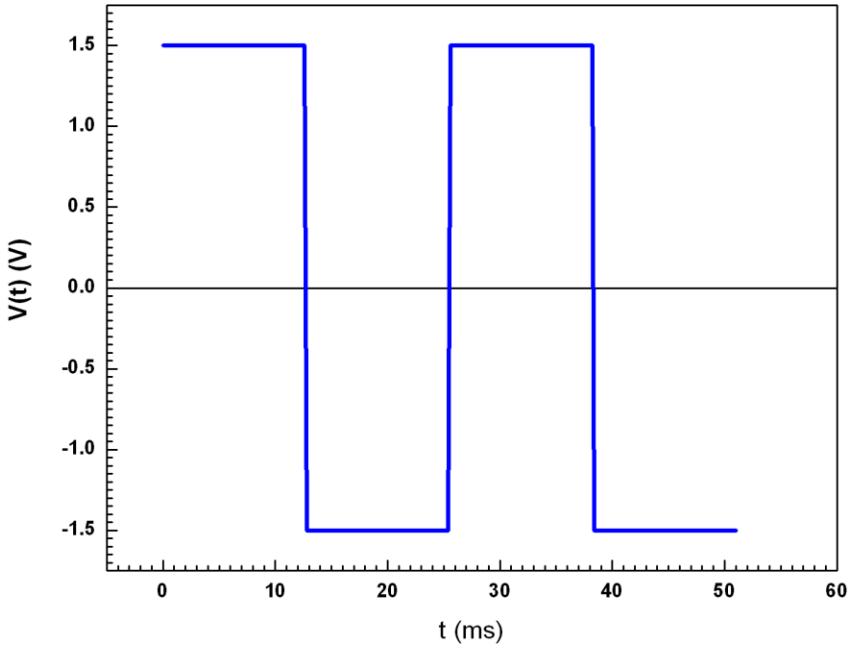


in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

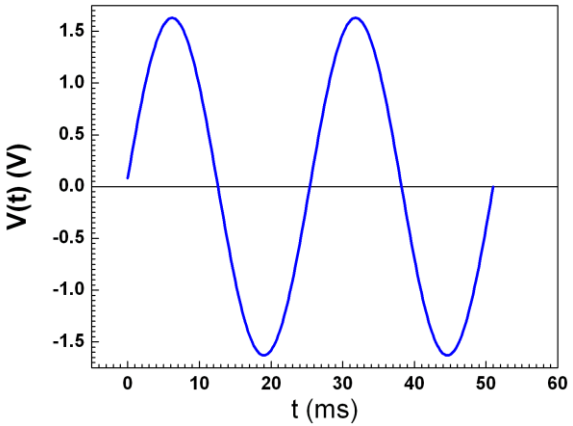
Jean Baptiste Joseph Fourier (1768 – 1830)

Let we try to create the square wave as a sum of sine waves of different frequencies

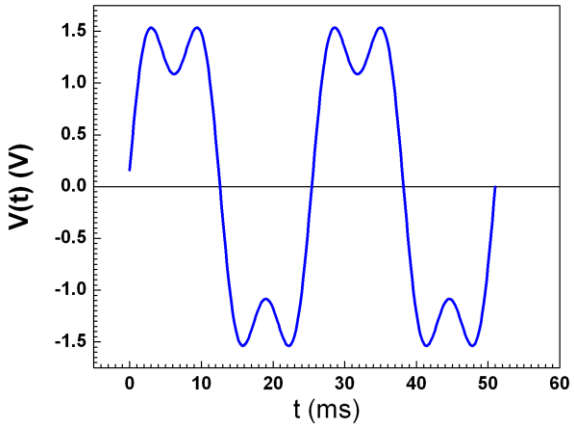
Square wave. F=40Hz, A=1.5V



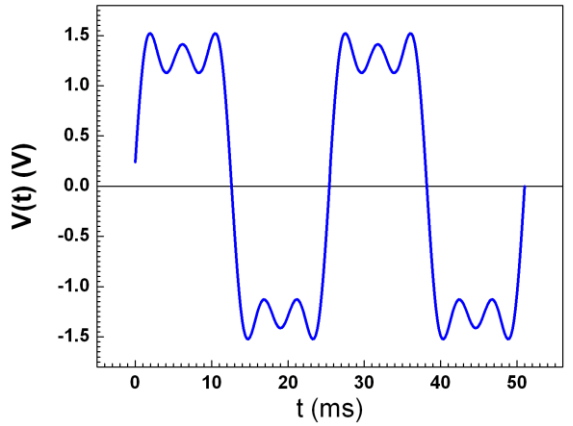
$A_1\sin(2\pi\omega t)$



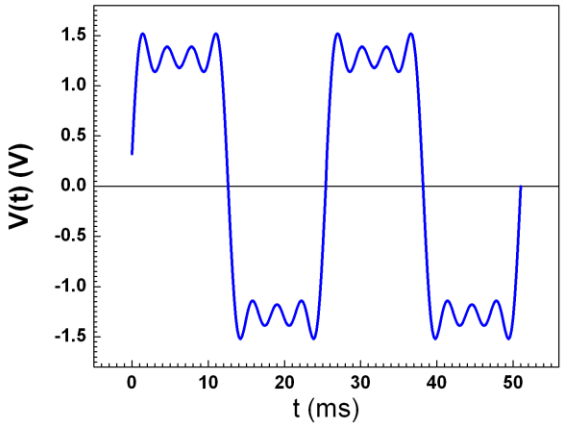
$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3)$



$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3) + A_5\sin(2\pi 5\omega t + \varphi_5)$



$A_1\sin(2\pi\omega t) + A_3\sin(2\pi 3\omega t + \varphi_3) + A_5\sin(2\pi 5\omega t + \varphi_5) + A_7\sin(2\pi 7\omega t + \varphi_7)$



Fourier Transform

The continues Fourier transformation of the signal h(t) can be written as:

$$H(f)=\int_{-\infty}^{+\infty} h(t)e^{2\pi jft}dt; \quad j=\sqrt{-1}$$

H(f) represents in frequency domain mode the time domain signal h(t)

Equation for inverse Fourier transform gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

$$h(t)=\int_{-\infty}^{+\infty} H(f)e^{-2\pi jft}df$$

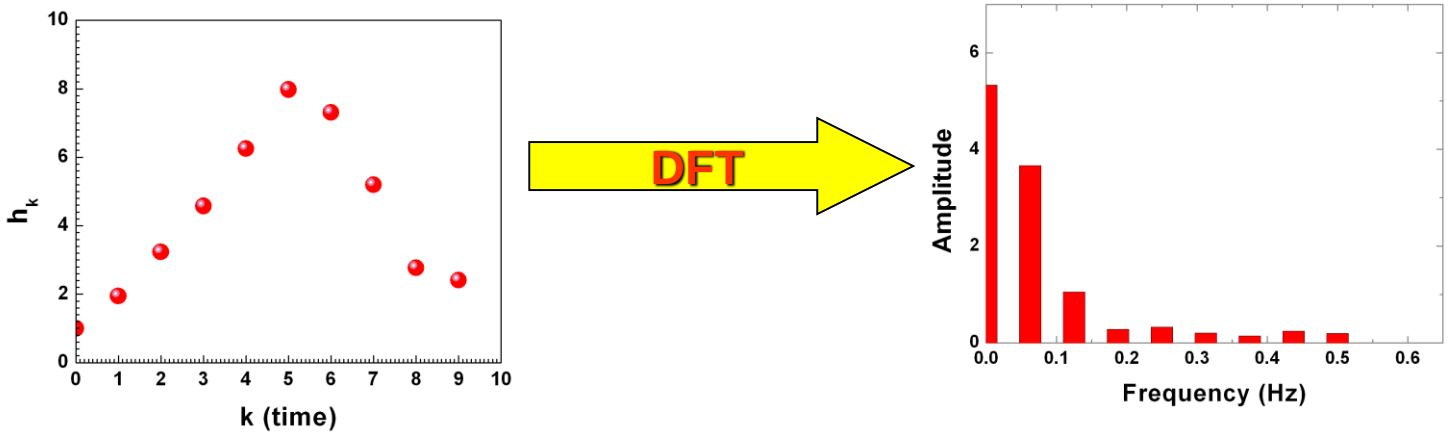
In real life we working with discrete representation of the time domain signal recorded during a finite time.

Discrete Fourier Transform

It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal h_k as sum of the frequency harmonic calculated as:

$$H_n = H(f_n) = \frac{1}{N}\sum_{k=0}^{N-1} h_k e^{2\pi kn/N}$$

Δ is the sampling interval, N – number of collected points



Time domain **Frequency domain**

Discrete Fourier Transform

For periodic signals with period T_0:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

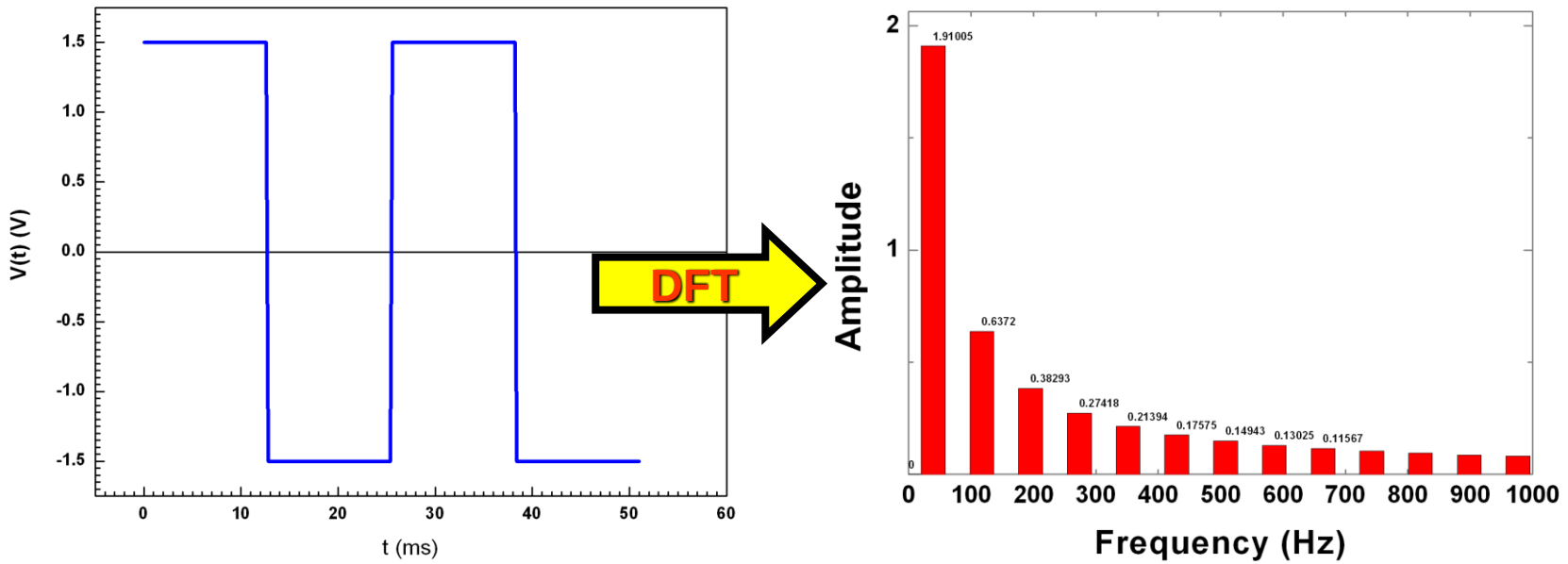
$$a_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt; \qquad b_n = \frac{2}{T_0}\int_0^{T_0} F(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt;$$

$$a_0 = \frac{2}{T_0}\int_0^{T_0} F(t)dt;$$

Discrete Fourier Transform

Now how I found the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

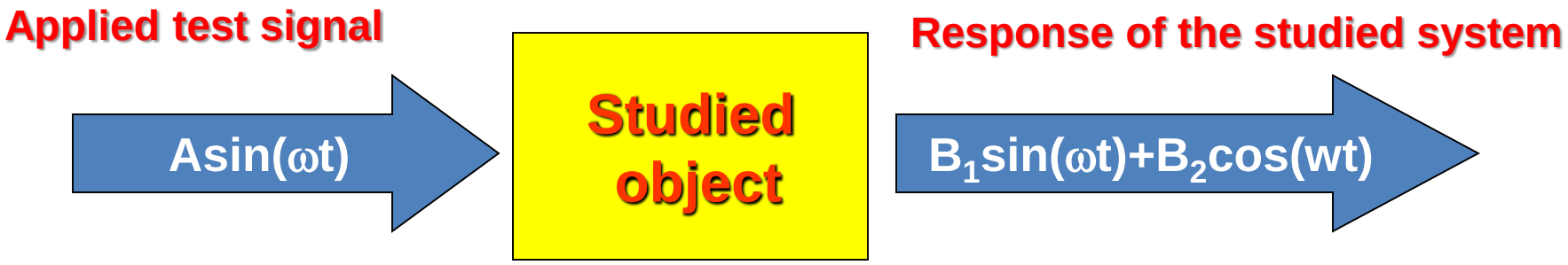
Time domain signal



Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.

Frequency Domain Spectroscopy

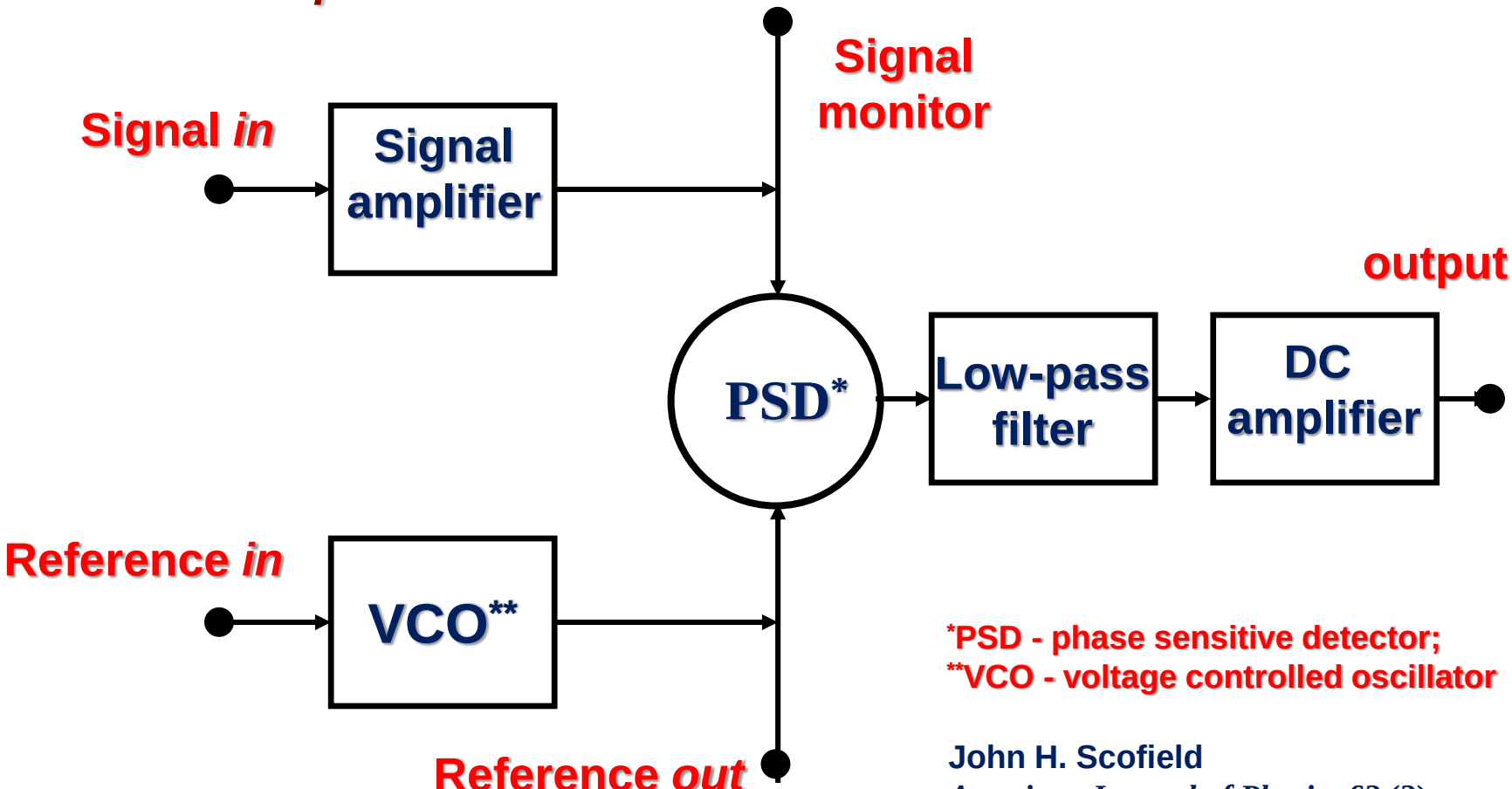
(linear system)



We applying the sine wave signal to the tested object and measuring the response. Varying the frequency we can study the frequency properties of the system.

Lock-in amplifier

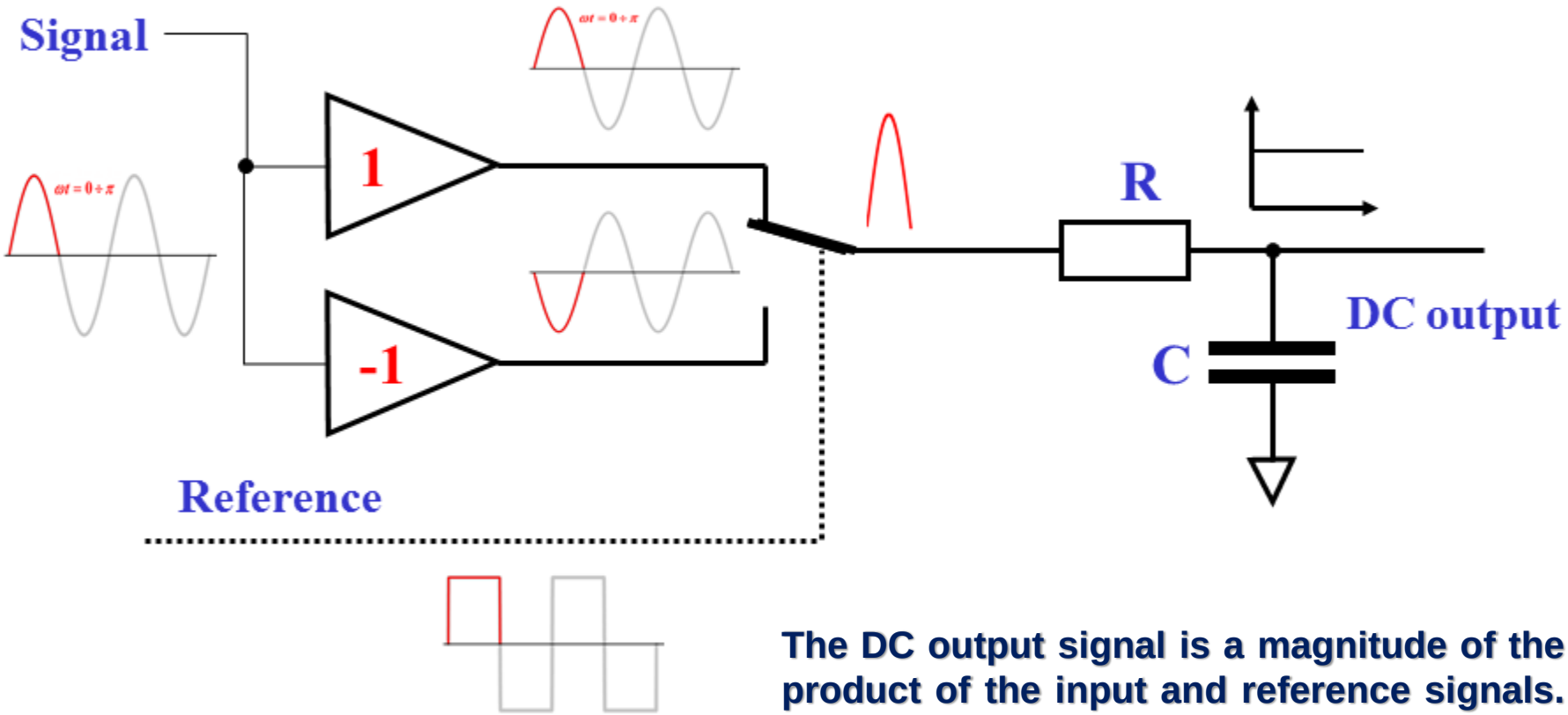
Now about the most powerful tool which can be used in frequency domain technique.



*PSD - phase sensitive detector;
**VCO - voltage controlled oscillator

John H. Scofield
American Journal of Physics 62 (2) 129-133 (Feb. 1994).

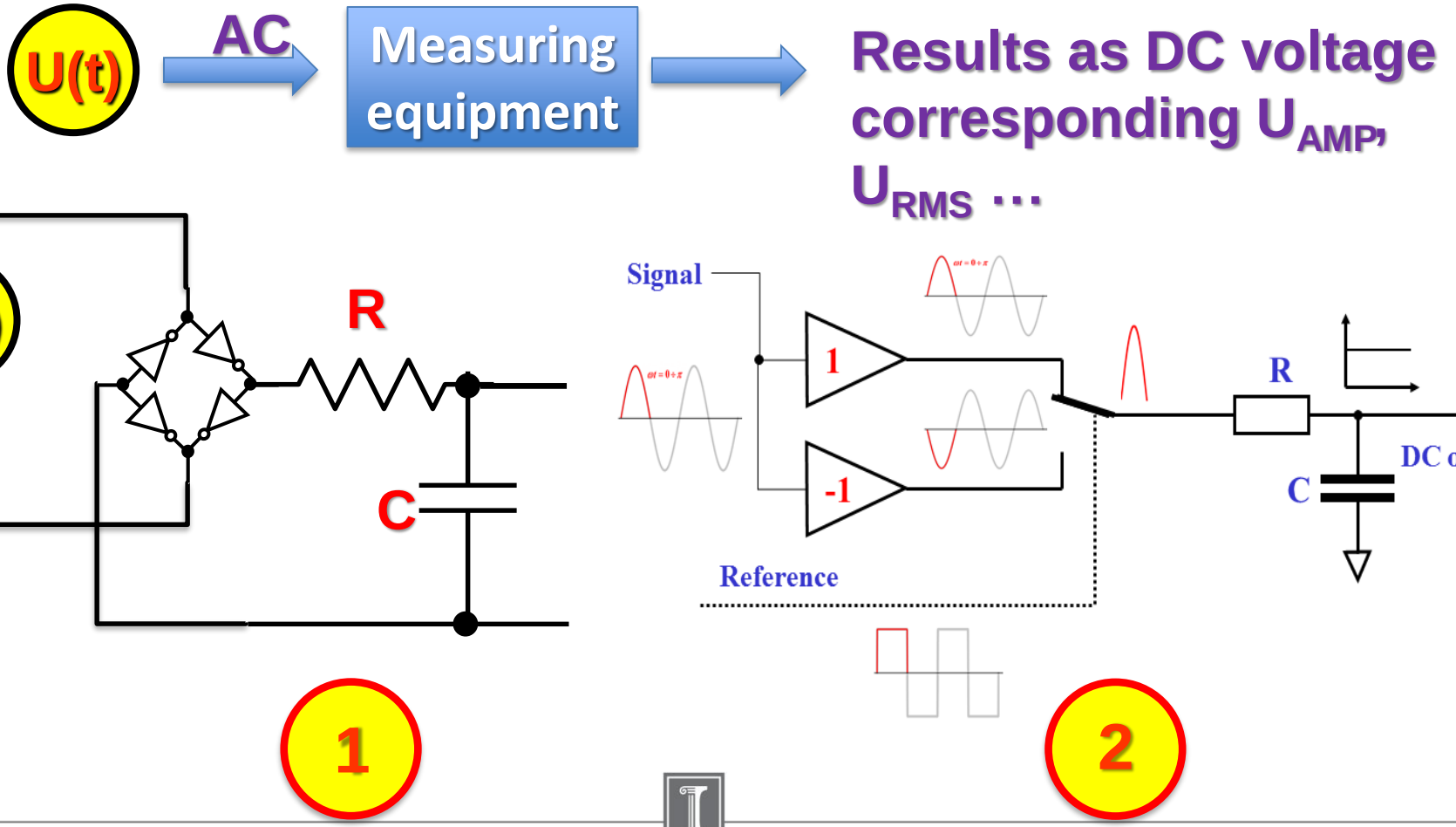
Lock-in amplifier. How it works.



The DC output signal is a magnitude of the product of the input and reference signals. AC components of output signal are filtered out by the low-pass filter with time constant τ (her τ=RC)

Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

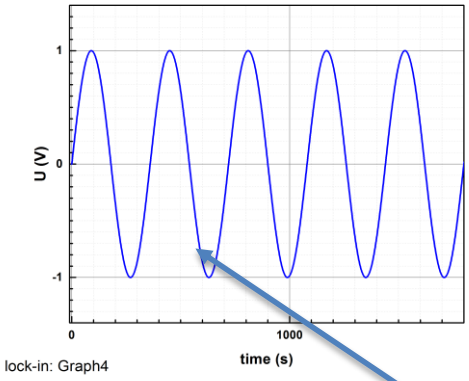
DMM, lock-in etc.

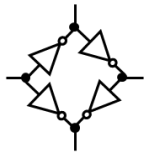


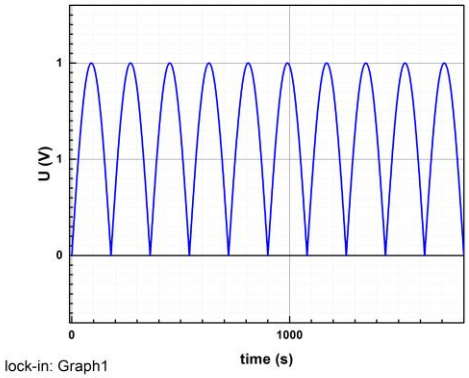
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

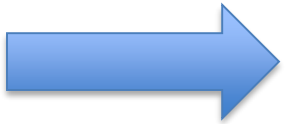
1

Clean sine wave – no "noise"









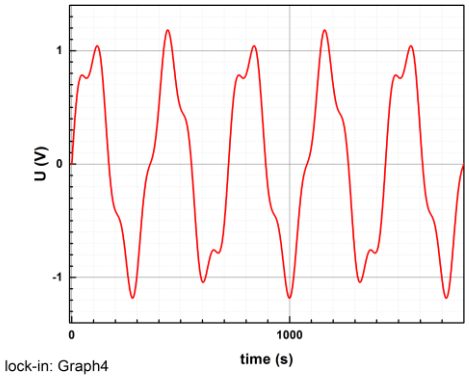
U_{DC}=0.63643V

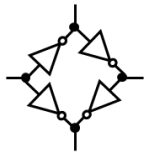
We need to measure the amplitude/rms value of the sine wave

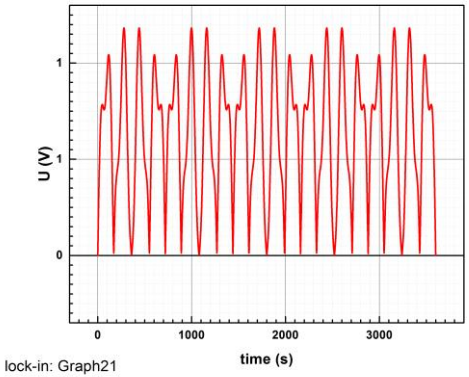
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

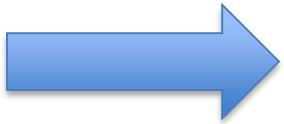
1

"Noisy" sine wave









U_{DC}=0.64208V

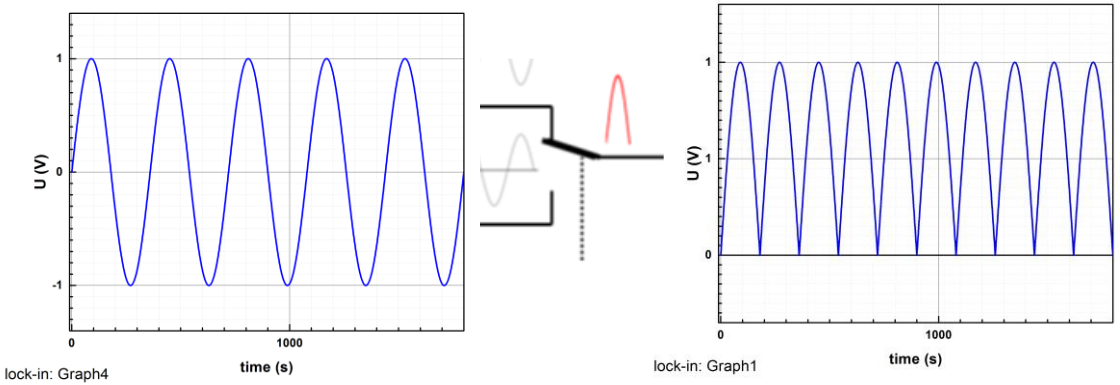
compare to

U_{DC}=0.63643V

Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

2

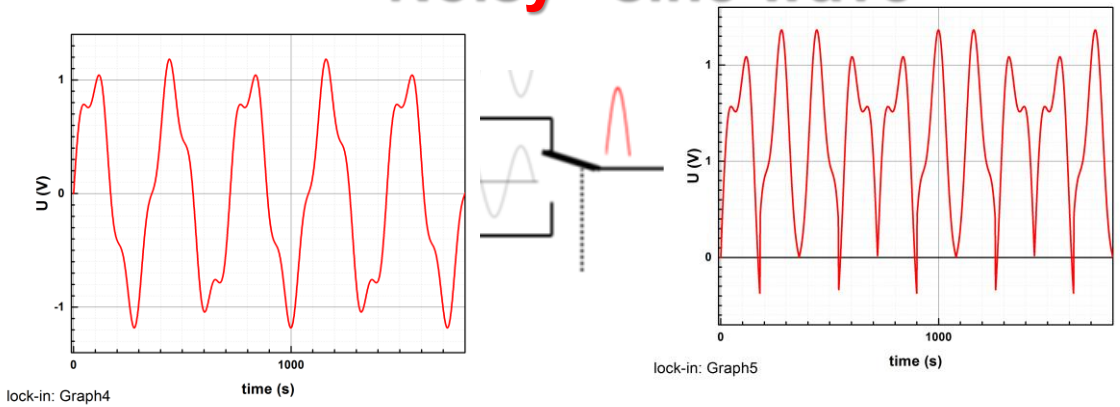
Clear sine wave – no "noise"



U_{DC}=0.63643V

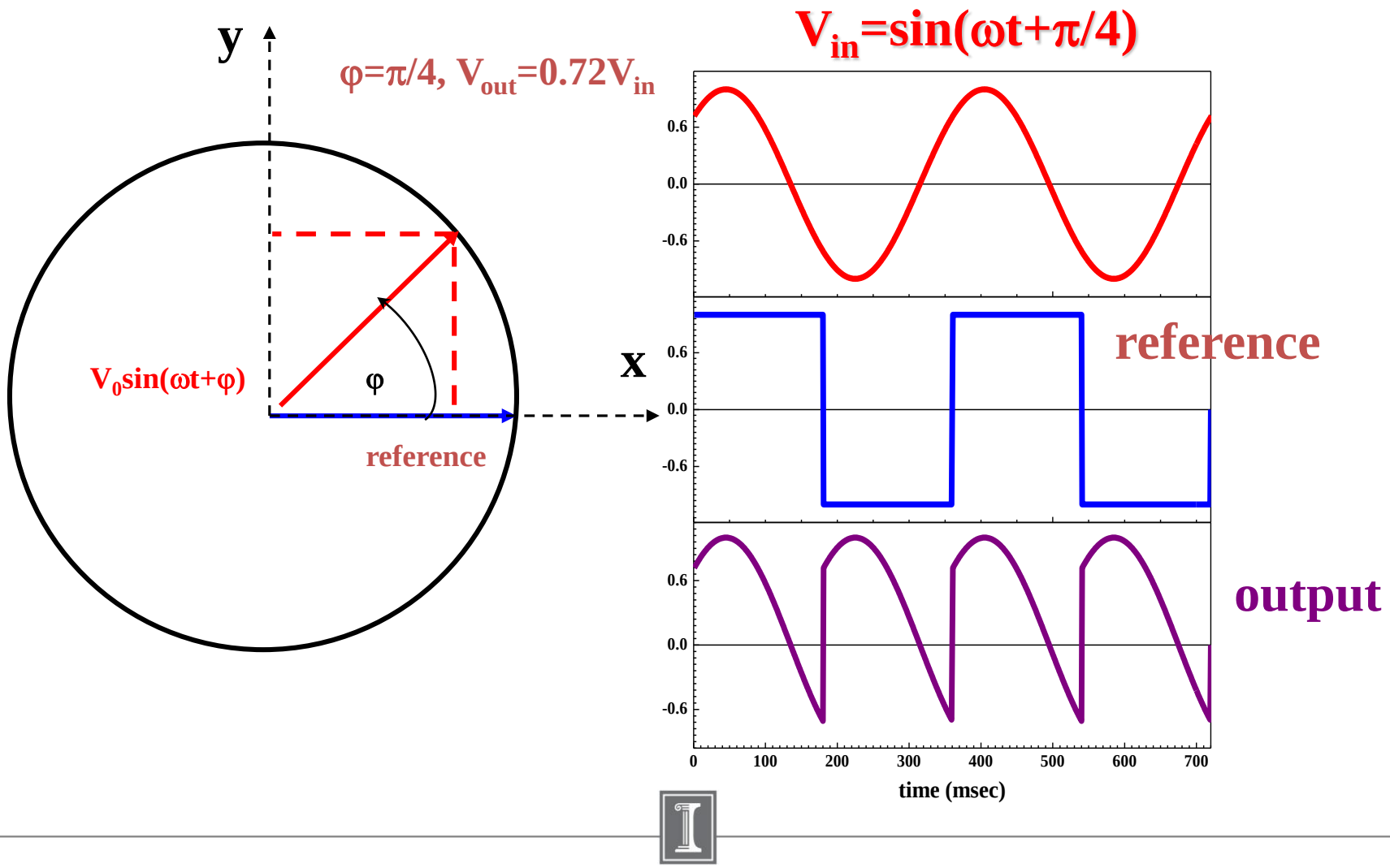
=

"Noisy" sine wave



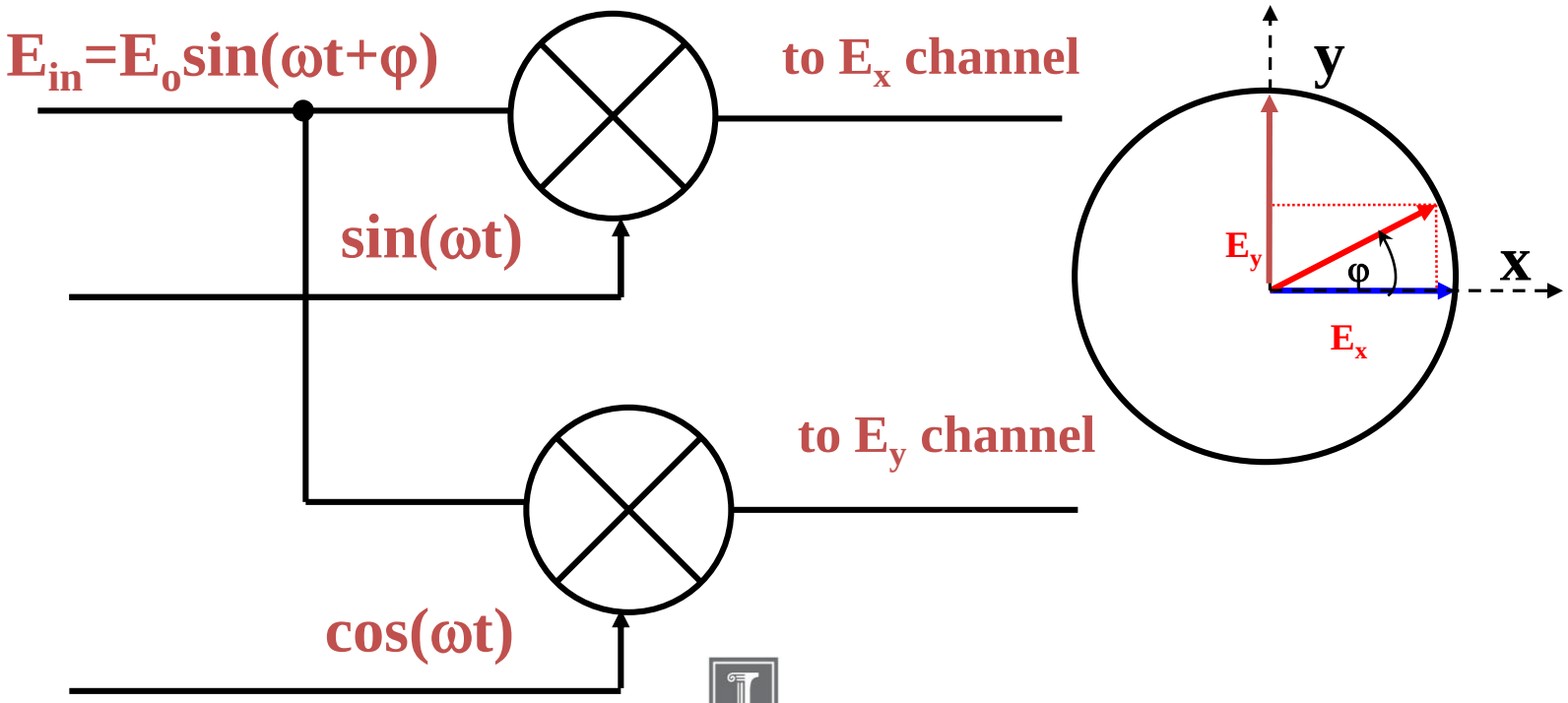
U_{DC}=0.63643V

Lock-in Amplifier. Phase shift.

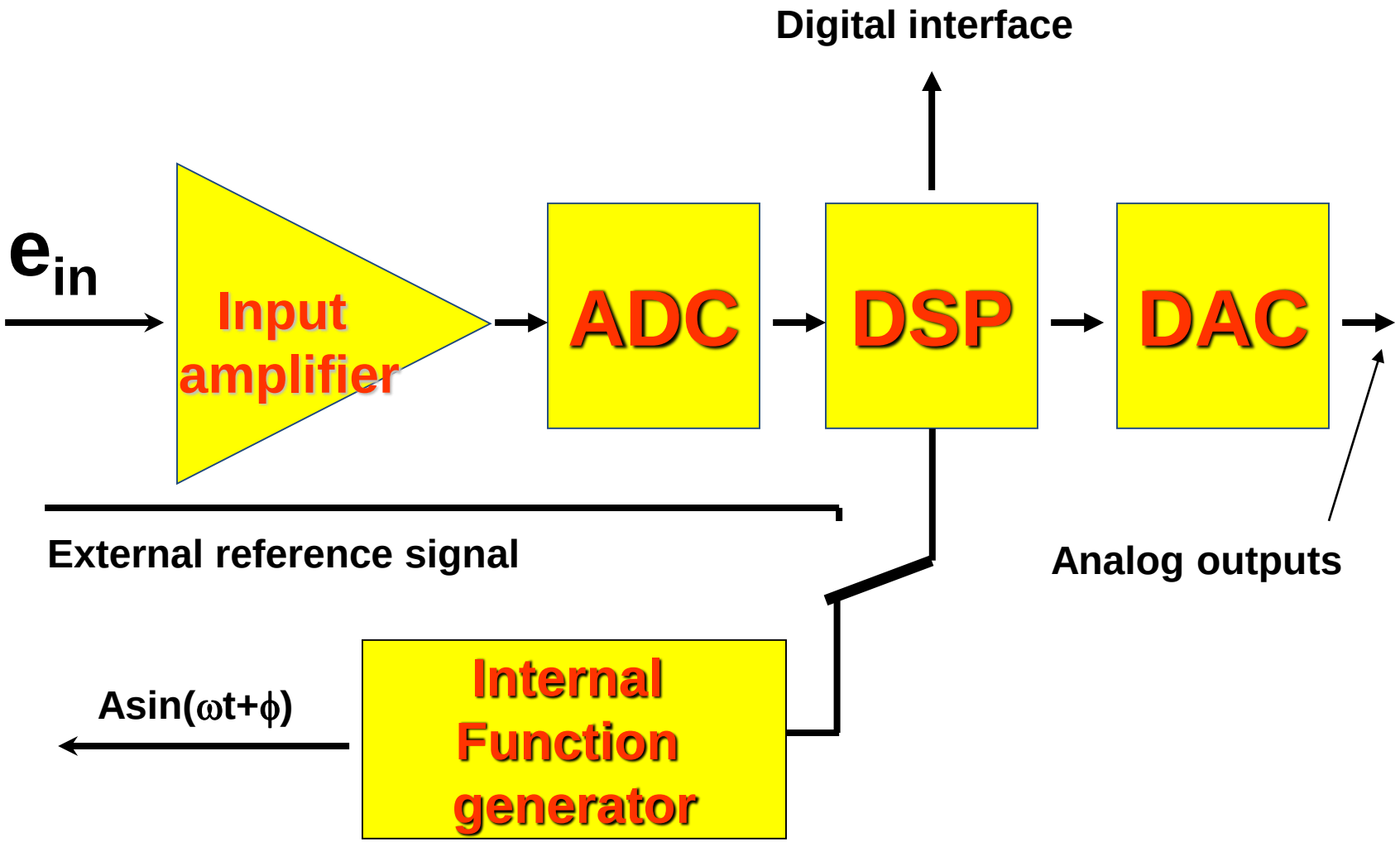


Lock-in Amplifier. Two Channels Demodulation.

In many scientific applications it is a great advantage to measure both components (E_x, E_y) of the input signal. We can use two lock-ins to do this or we can measure these value in two steps providing the phase shift of reference signal 0 and $\pi/2$. Much better solution is to use the lock-in amplifier equipped by two demodulators.



Digital Lock-in Amplifier



SR830. Digital Lock-in Amplifier

In SR830 manual you can find the chapter dedicated to general description of the lock-in amplifier idea

SR830 BASICS

WHAT IS A LOCK-IN AMPLIFIER?

Lock-in amplifiers are used to detect and measure very small AC signals - all the way down to a few nanovolts! Accurate measurements may be made even when the small signal is obscured by noise

experiment at the reference frequency. In the diagram below, the reference signal is a square wave at frequency ω_r. This might be the sync output from a function generator. If the sine output from

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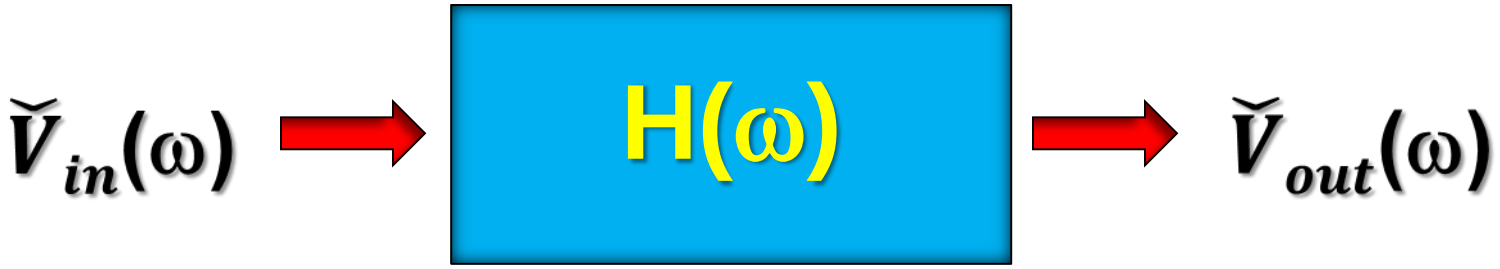
Digital Lock-in amplifier. SR830

Experiments. Main idea. Investigating the frequency response of circuit.



Frequency domain representation of the system

Response function → $\breve{H}(\omega) = \dfrac{\breve{V}_{out}(\omega)}{\breve{V}_{in}(\omega)}$

Linear systems are those that can be modeled by linear differential equations.

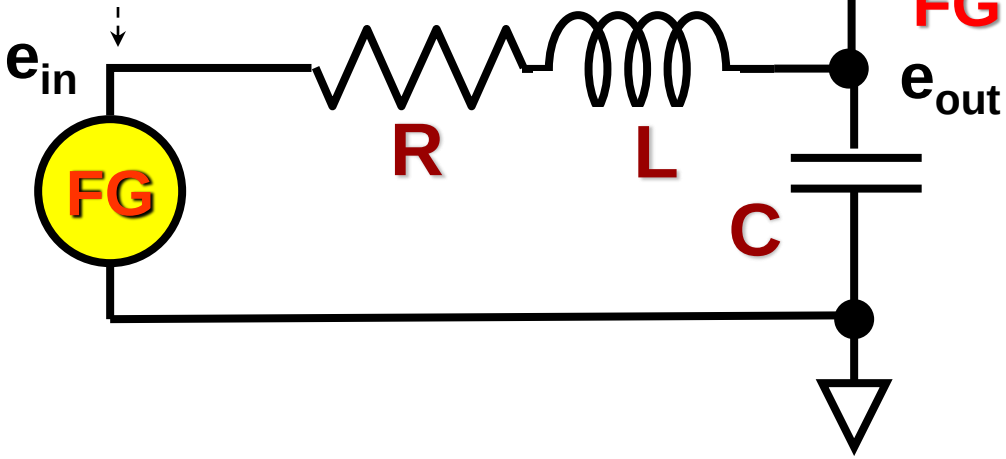
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit

SR830

GPIB

input

FG output

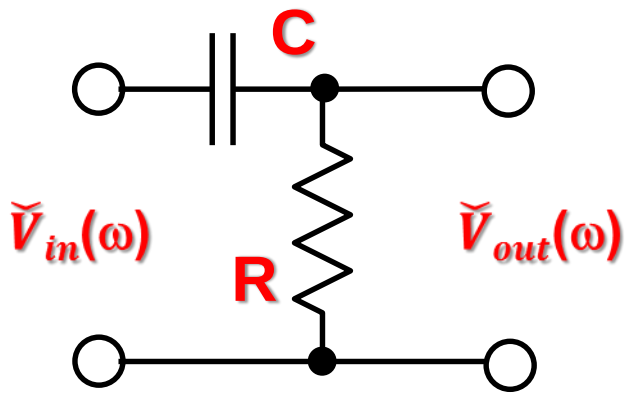


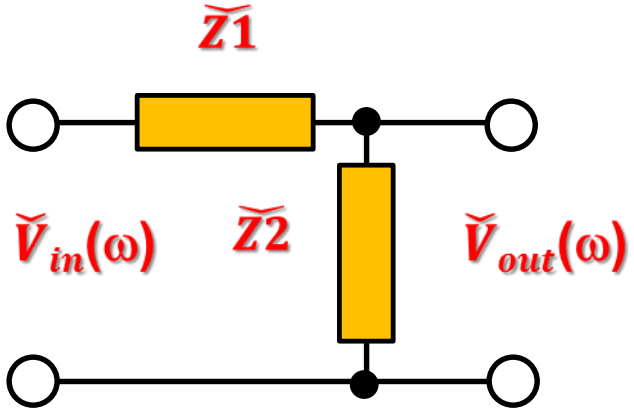
Setup for measurement of the transfer function of the RLC circuit.

$$h(\omega) = \frac{e_{out}}{e_{in}}$$

Experiments. Main Idea. Calculation of the Response Function in Frequency Domain Mode.

Example 1. High-pass filter.

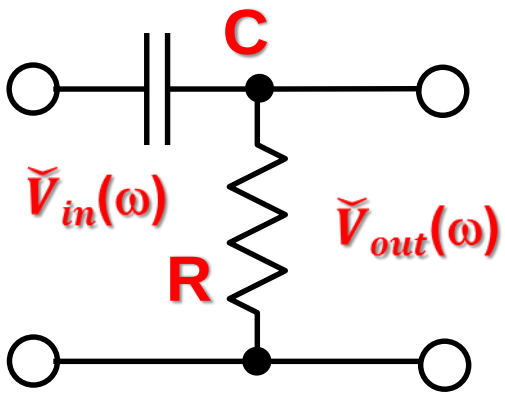




Applying the Kirchhoff Law to this simple network

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega) \frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter



Ideal case

$$\tilde{Z}_R = R$$

$$\tilde{Z}_L = j\omega L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

More realistic

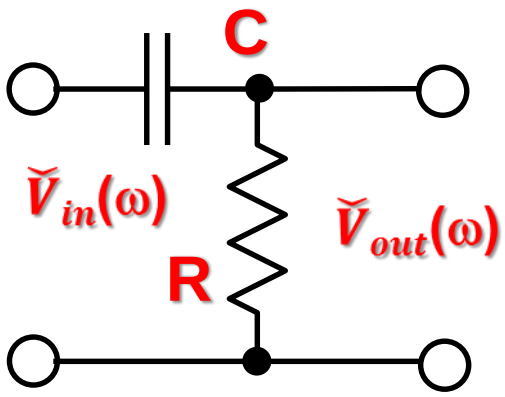
$$\tilde{Z}_R = R + \ldots$$

$$\tilde{Z}_L = j\omega L + R_L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = \frac{1}{j\omega C + R_C^{-1}}$$

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega) \frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter



τ – time constant of the filter

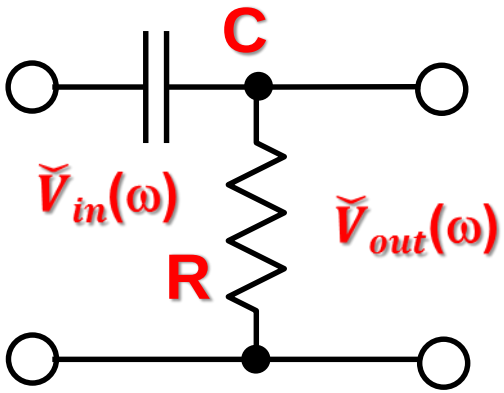
ω_C - cutoff frequency

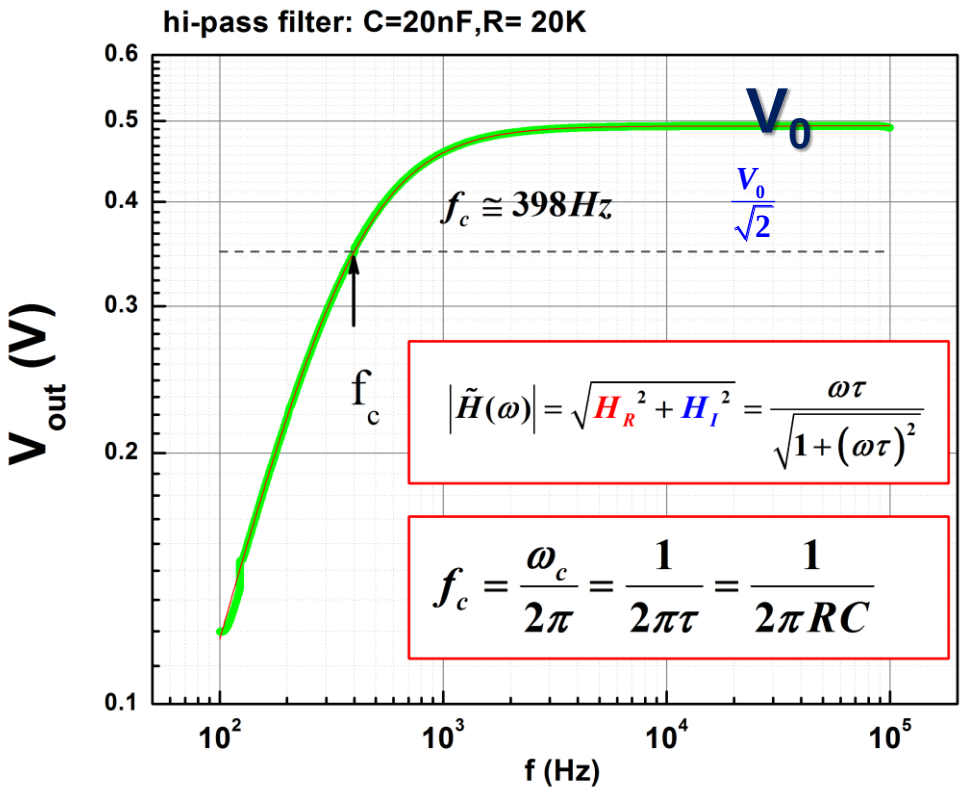
$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{\omega\tau}{\left(1 + \omega^2\tau^2\right)}\left(\omega\tau + j\right);$$

where $\tau = RC = \omega_c^{-1}$;

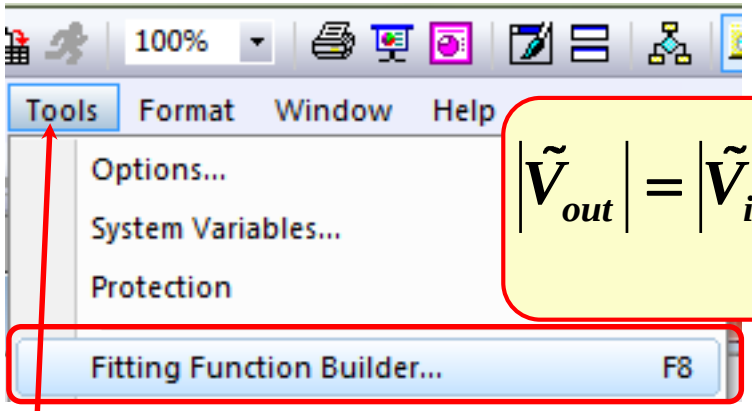
$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R{}^2 + H_I{}^2} = \frac{\omega\tau}{\sqrt{1 + \left(\omega\tau\right)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = \arctan\left(\frac{1}{\omega\tau}\right)$$

Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter





High-pass Filter. Fitting.



$$\left|\tilde{V}_{out}\right| = \left|\tilde{V}_{in}\right| * \left|\tilde{H}(\omega)\right| = V_0 * \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}; \quad \tau = RC$$

Fitting parameters: V_0, τ, Voff

V0,tau,Voff

Parameters | Constants

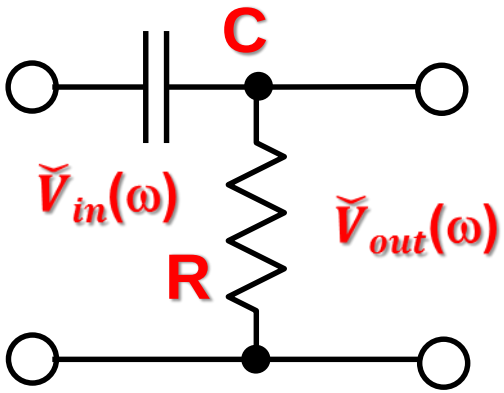
Param	Unit	Meaning	Fixed	Initial Value	Significant Digits
V0		?	☐	1	System
tau		?	☐	1	System
Voff		?	☐	1	System

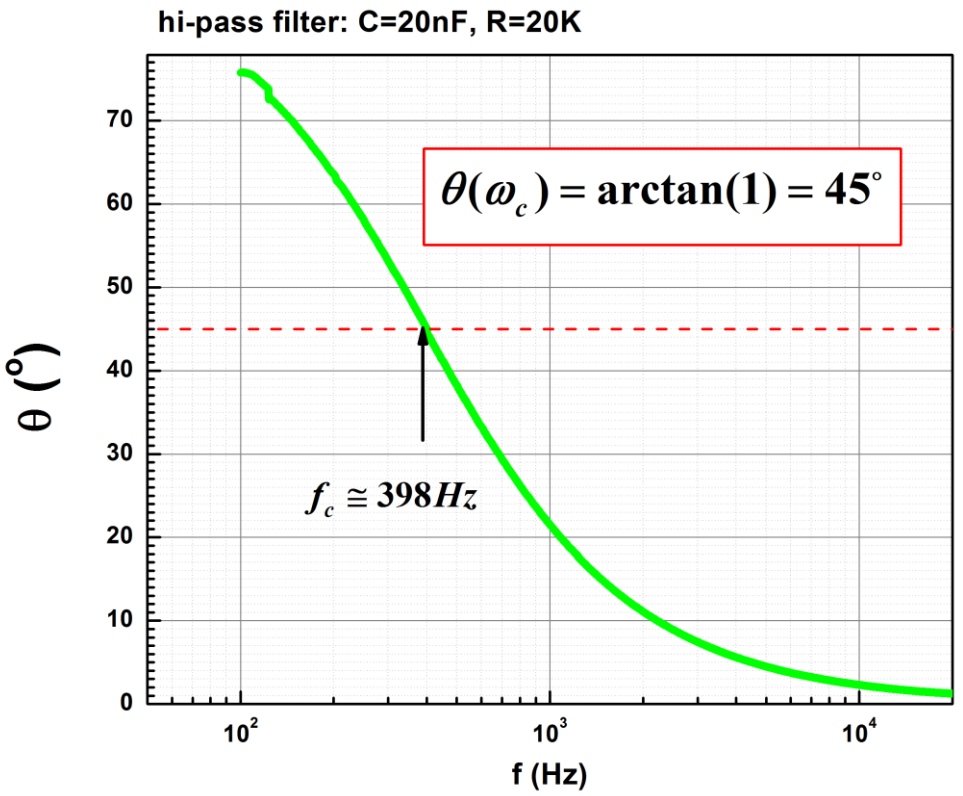
Fitting function ➡

Function Body (Dependent Variables : y)

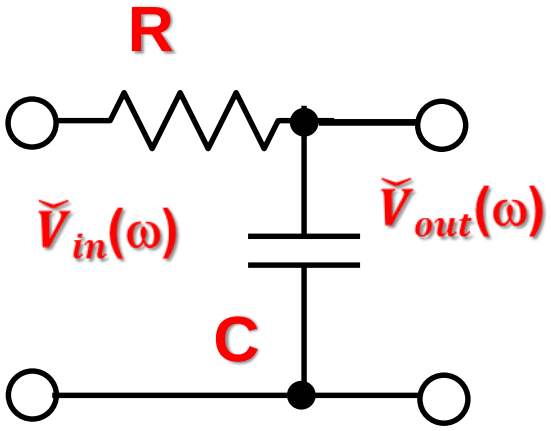
```
y=V0*2*pi*x*tau/sqrt(1+(2*pi*tau)^2)+Voff
```

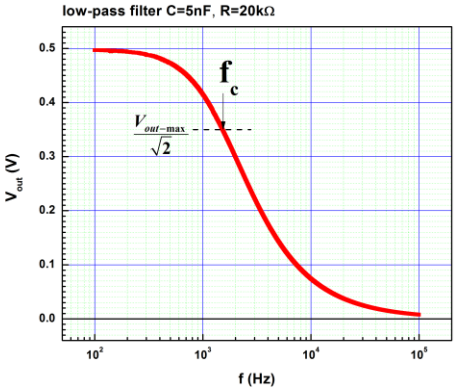
Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter.

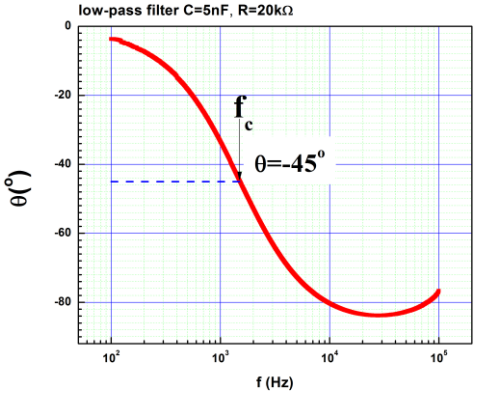




Experiments. Calculation of the Response Function in Frequency Domain Mode. Low-pass Filter





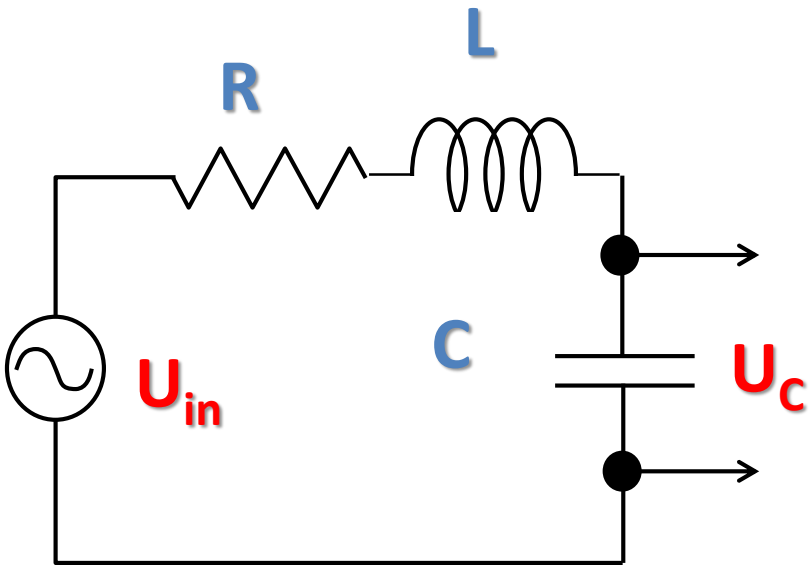


$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{(1 - j\omega\tau)}{(1 + \omega^2\tau^2)};$$

where $\tau = RC = \omega_c^{-1}$;

$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R{}^2 + H_I{}^2} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = -\arctan(\omega\tau)$$

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .



$$U_C = U_{in} \bullet \frac{Z_C}{Z_C + Z_L + R} =$$

$$= \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R}$$

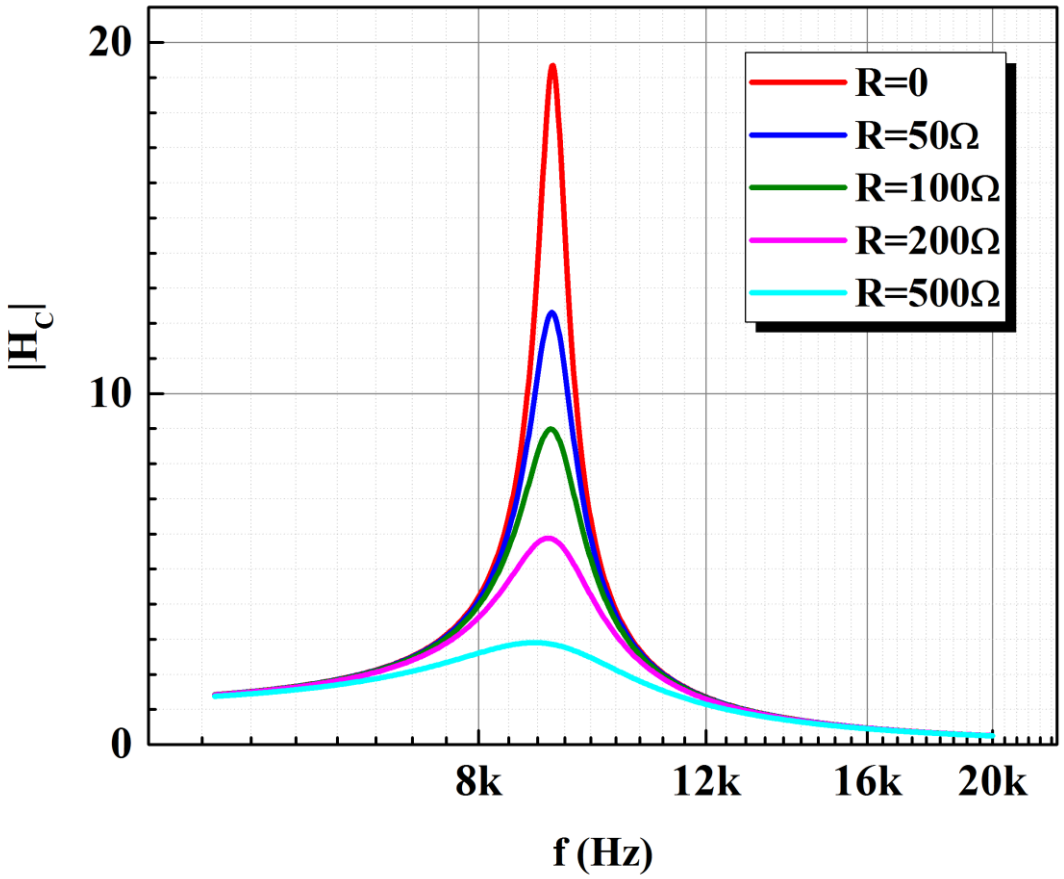
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .

$$H = \frac{U_C}{U_{in}} = \frac{1}{(1-\omega^2 LC) + j\omega CR} = \frac{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right) - j\omega CR}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \omega^2 C^2 R^2} \times;$$

$$\omega_0 = \frac{1}{\sqrt{LC}}; \nu \equiv \frac{\omega}{\omega_0}; Q = \frac{1}{R}\sqrt{\frac{L}{C}};$$

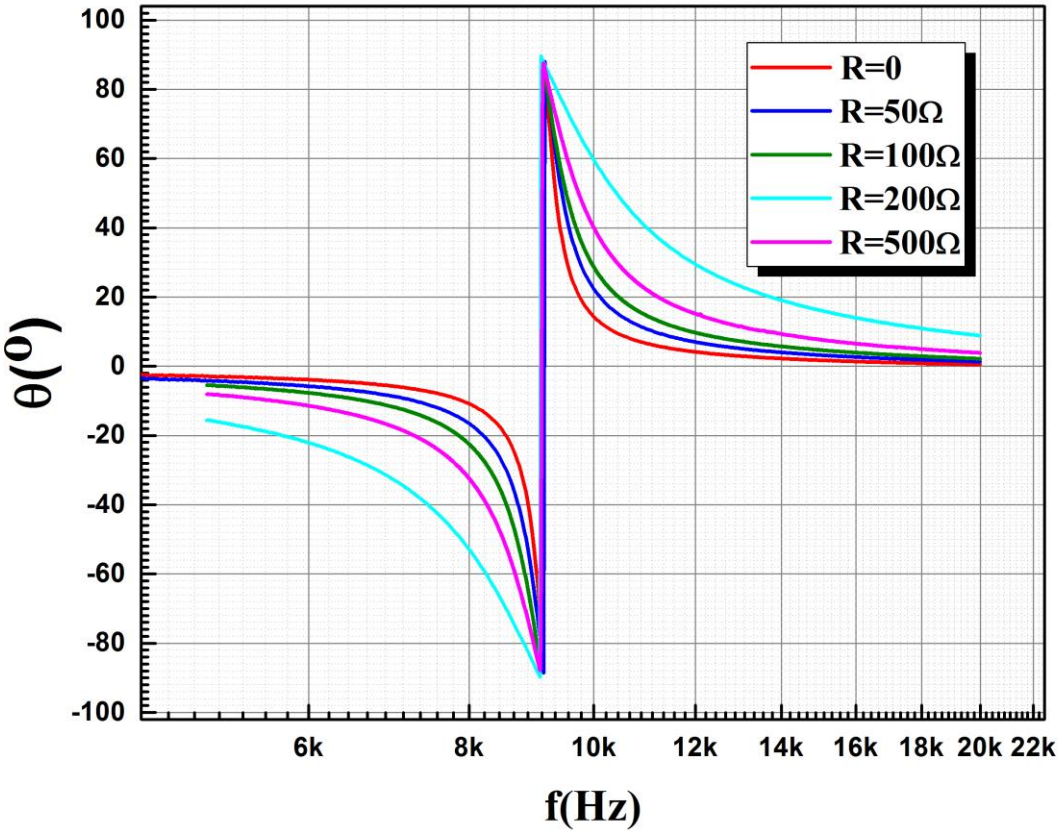
$$H = \frac{(1-\nu^2) - j\frac{\nu}{Q}}{(1-\nu^2)^2 + \frac{\nu^2}{Q^2}}; \theta = -\tan^{-1}\left(\frac{\nu}{Q\left(1-\nu^2\right)}\right)$$

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit .



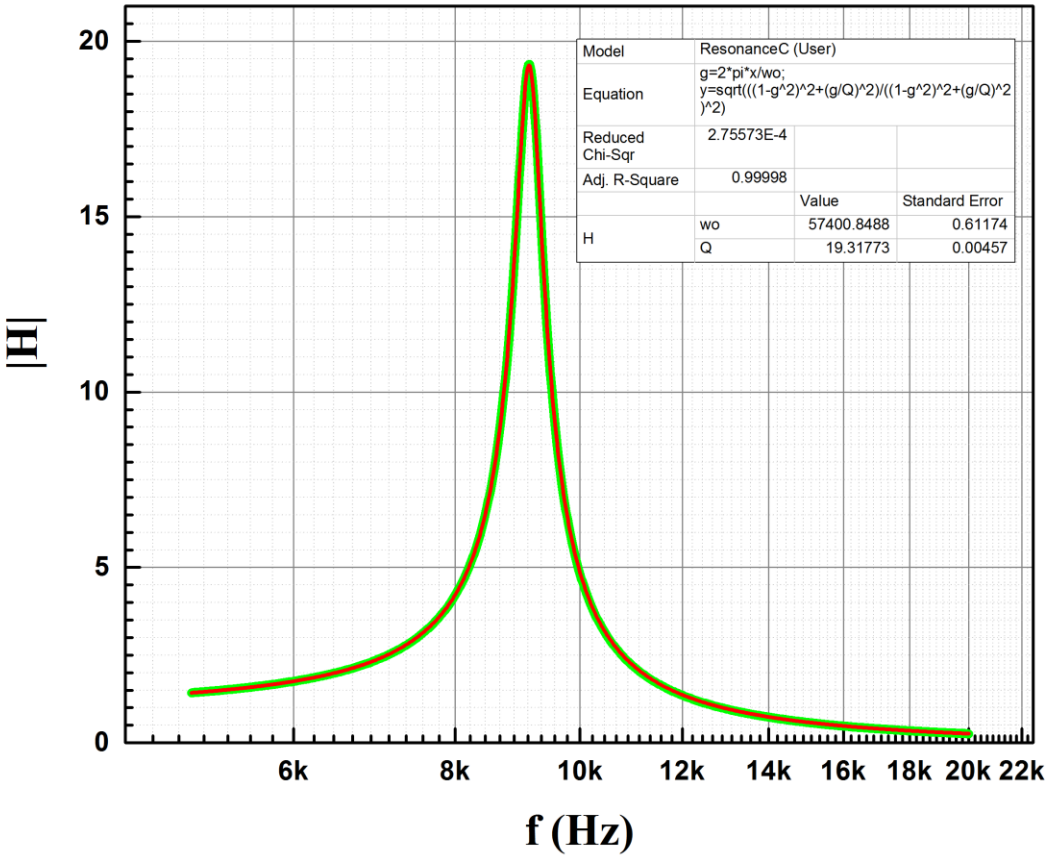
The resonance curves obtained on RLC circuits with different damping resistors.

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit



The resonance curves obtained on RLC circuits with different damping resistors

Fitting. RLC Resonance Circuit.

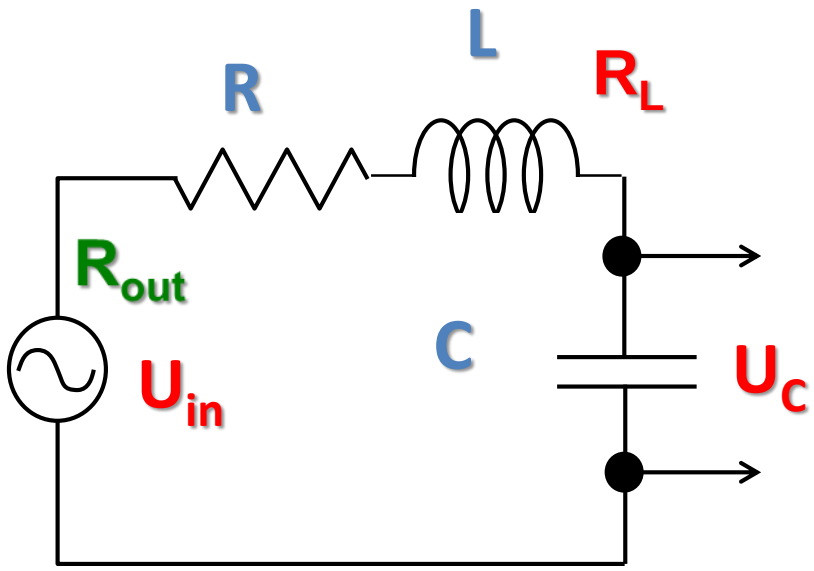


$$|H| = \frac{\sqrt{(1-\gamma^2)^2 + \left(\frac{\gamma}{Q}\right)^2}}{\left(1-\gamma^2\right)^2 + \left(\frac{\gamma}{Q}\right)^2}; \gamma = \frac{\omega}{\omega_0}$$

fitting function for |H|

variable parameters: ω_0 and Q

Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit

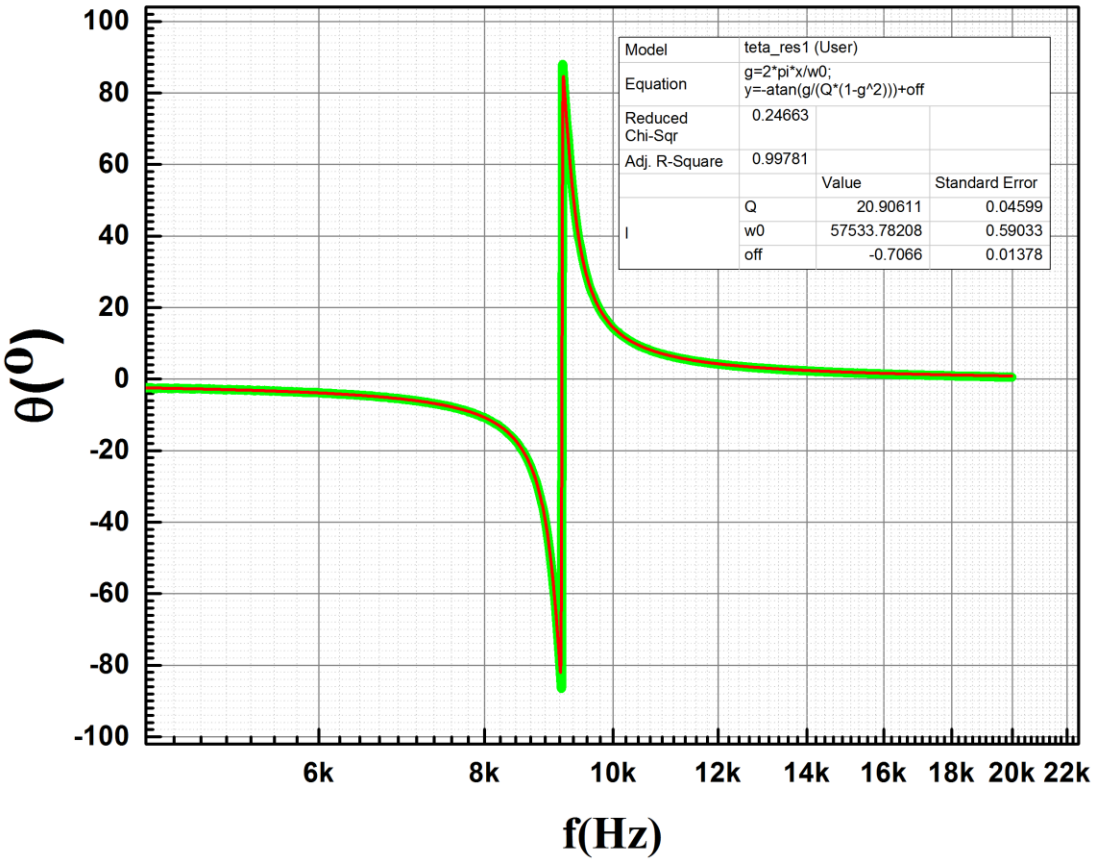


Actual damping resistance is a sum of R, R_L (resistance of the coil) and R_{out} (output resistance of the function generator)

R=0; R_L =35.8Ω; R_{out}=50Ω

Actual R calculated from fitting pars is~88.8Ω what is reasonable close to 85.8Ω

Fitting. RLC Resonance Circuit.



$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

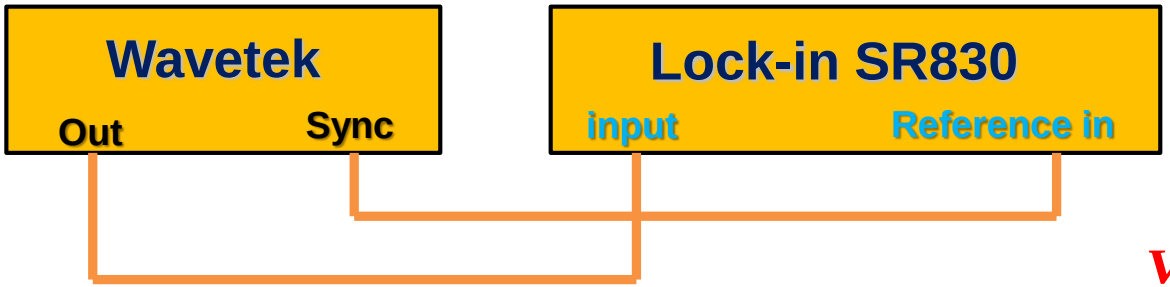
measured

$$\theta = -\tan^{-1}\left(\frac{\gamma}{Q\left(1-\gamma^2\right)}\right); \gamma = \frac{\omega}{\omega_0}$$

fitting function

variable parameters: ω_0 and Q

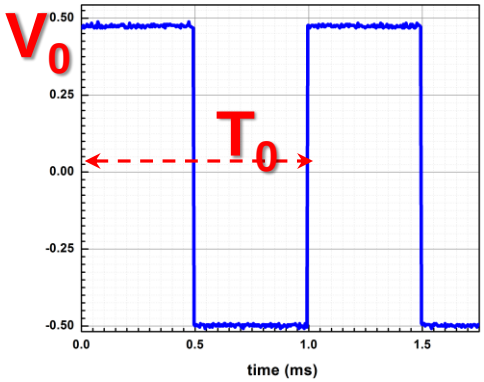
From Time Domain to Frequency Domain. Experiment.



$$V = V_0 \left(0 < t \le \frac{T_0}{2} \right);$$

$$-V_0 \left(\frac{T_0}{2} < t \le T_0 \right)$$

F(t) – periodic function $F(t)=F(t+T_0)$:



Time domain pattern

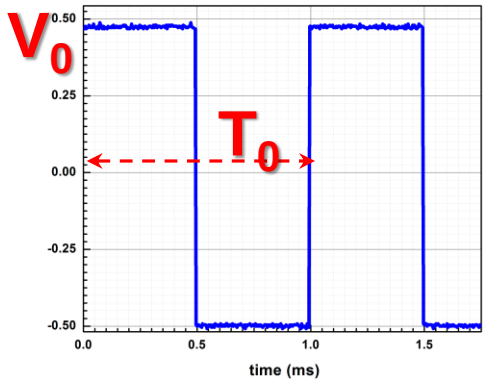
Frequency domain ?

$$a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos\left(\frac{2\pi n t}{T_0} \right) dt;$$

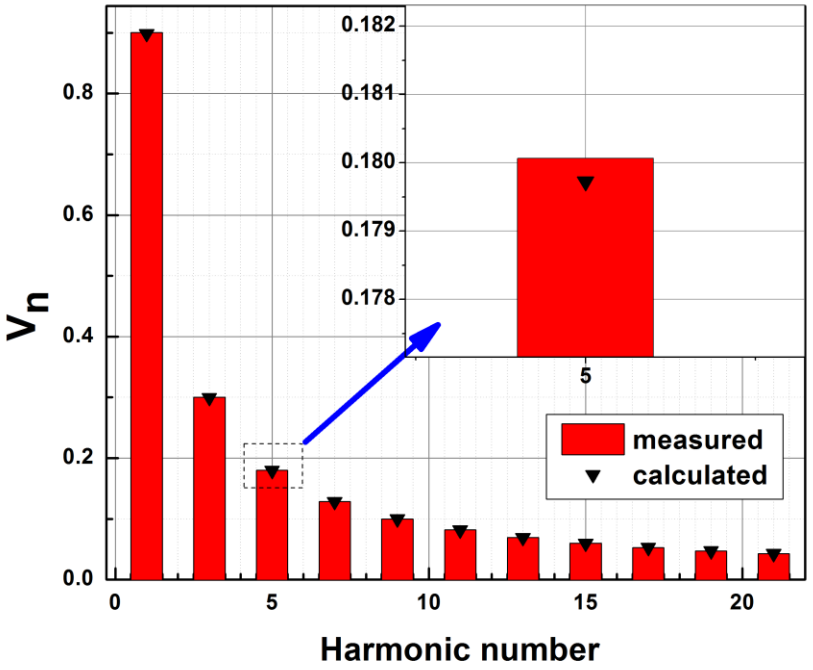
$$b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin\left(\frac{2\pi n t}{T_0} \right) dt;$$

$$a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt$$

From Time Domain to Frequency Domain. Experiment with SR830. Results.



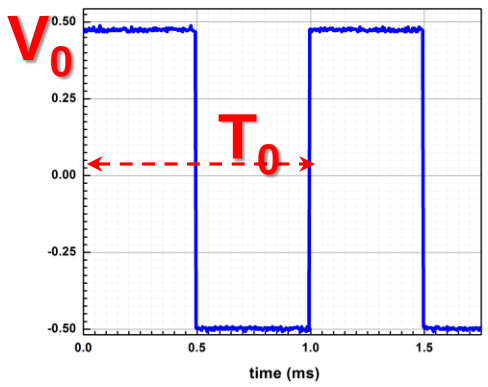
Time domain



Spectrum measured by SR 830 lock-in amplifier

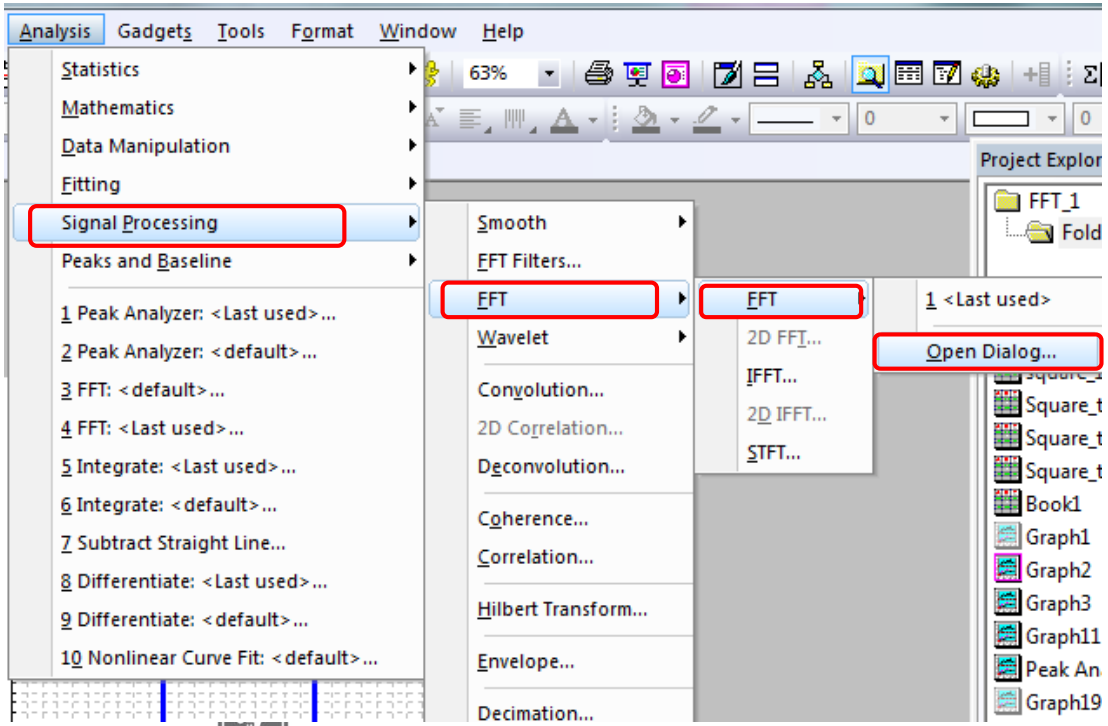
Frequency domain

From Time Domain To Frequency Domain. FFT using Origin. Results.

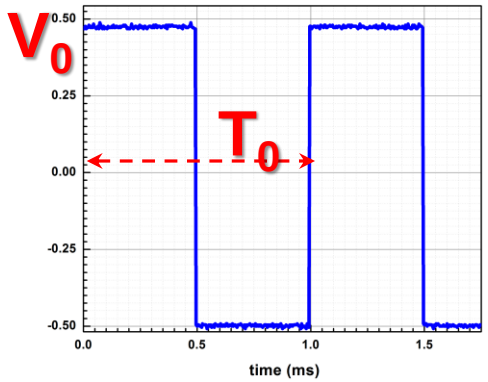


Time domain taken by Tektronix scope

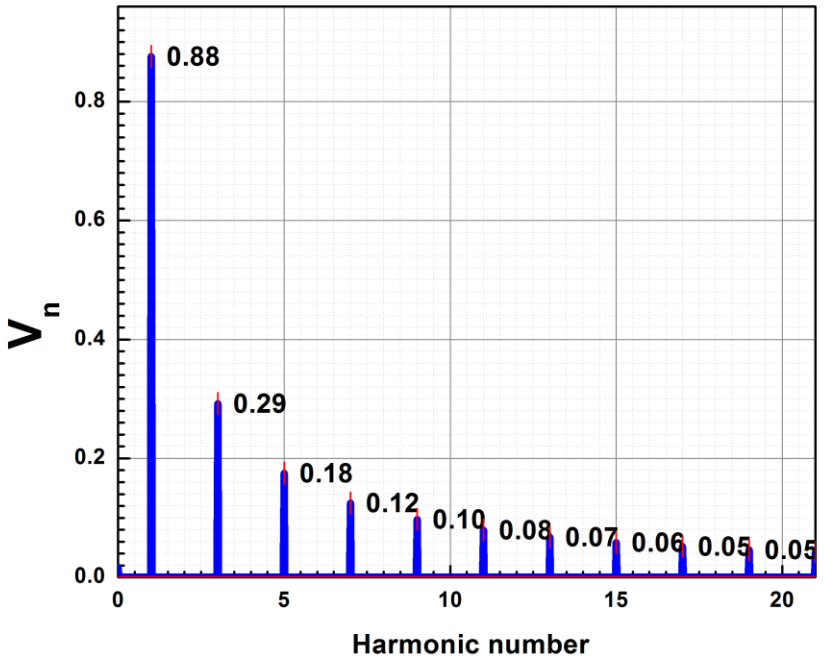
Data file can be used to convert time domain to frequency domain



From Time Domain to Frequency Domain. FFT using Origin. Results.

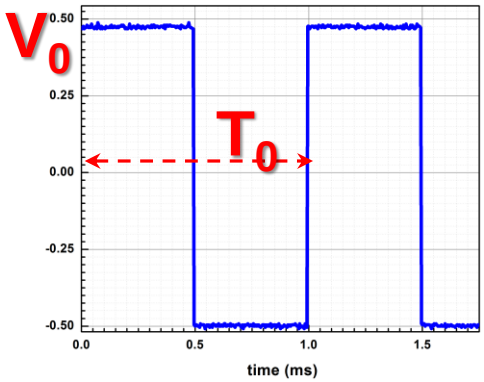


Time domain taken by Tektronix scope

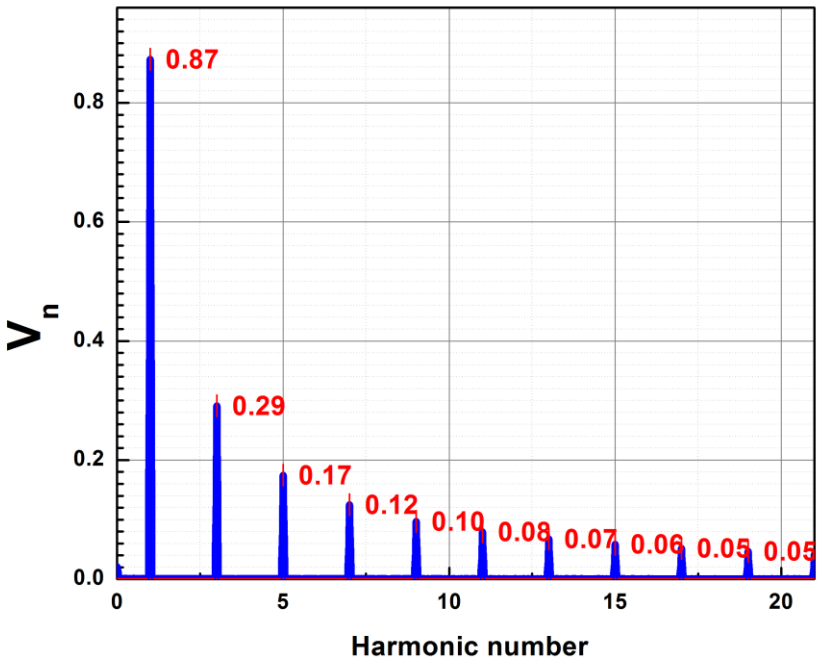


Spectrum calculated by Origin. Accuracy is limited because of the limited resolution of the scope

From Time Domain to Frequency Domain. Using of the Math Option of the Scope.

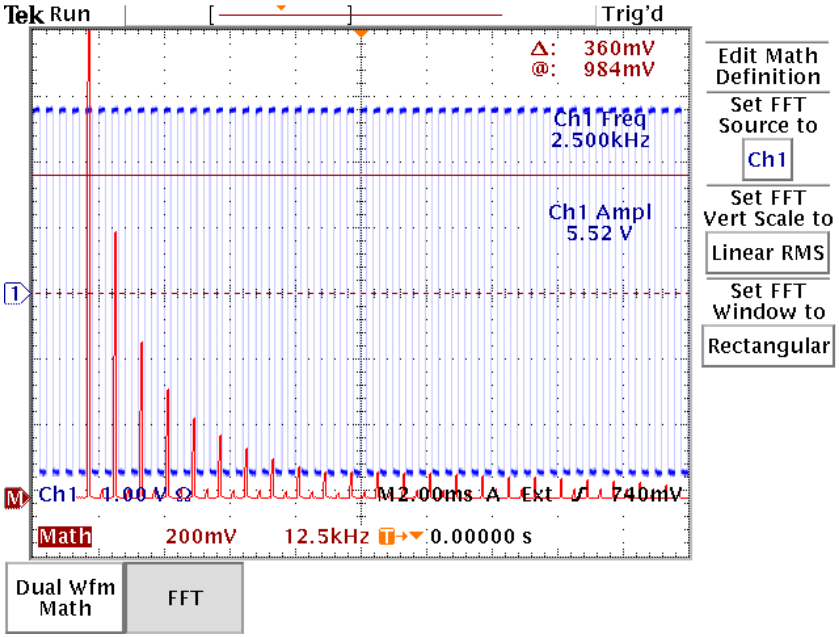


Time domain taken by Tektronix scope

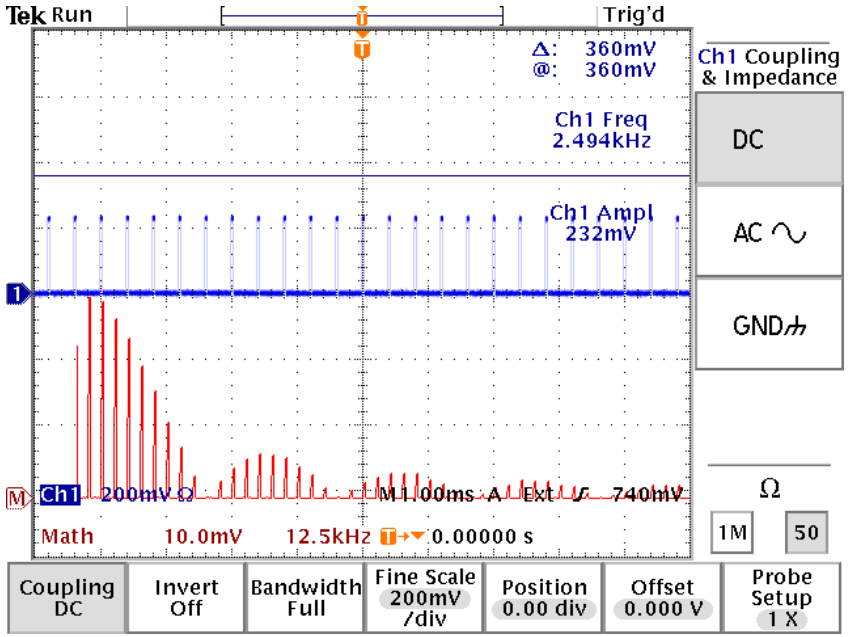


Spectrum calculated by Tektronix scope.
Accuracy is limited because of the limited resolution of the scope

From Time Domain to Frequency Domain. Using of the Math Option of the Scope.

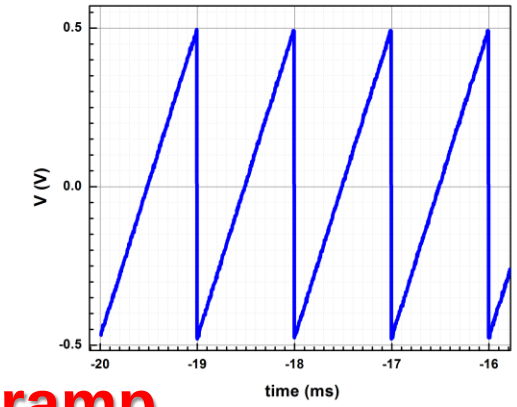


Spectrum of the square wave signal

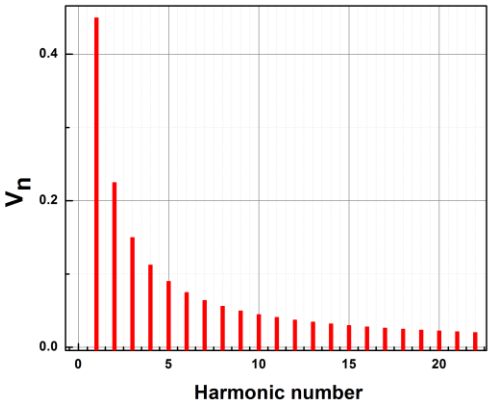


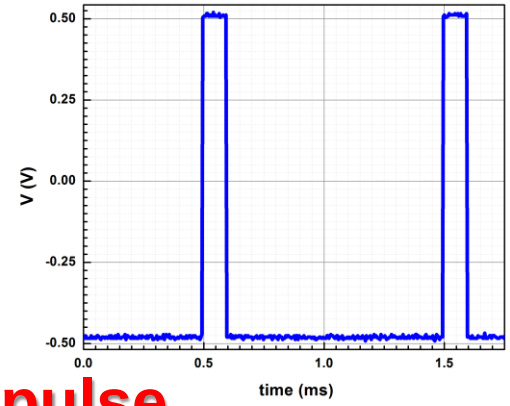
Spectrum of the pulse signal

From Time Domain to Frequency Domain. Different Waveforms. Using Lock-in.

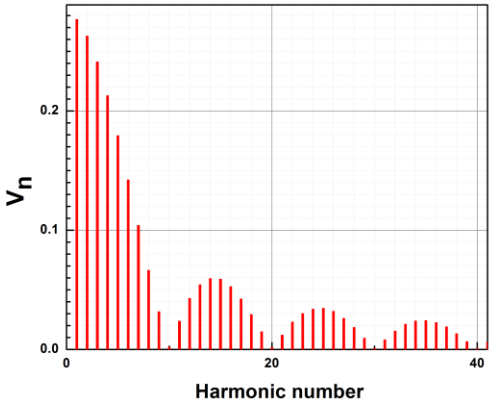


ramp



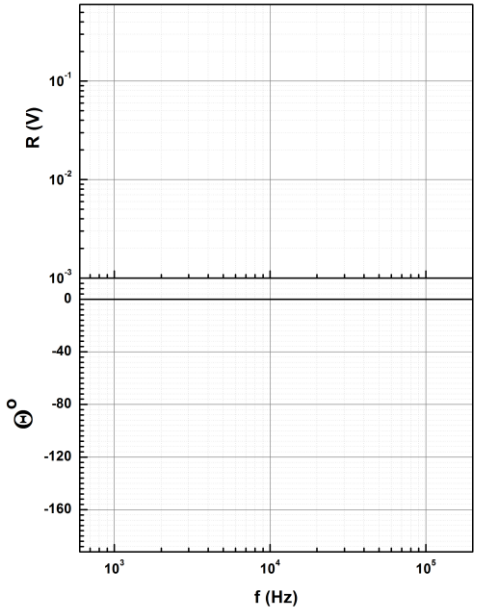


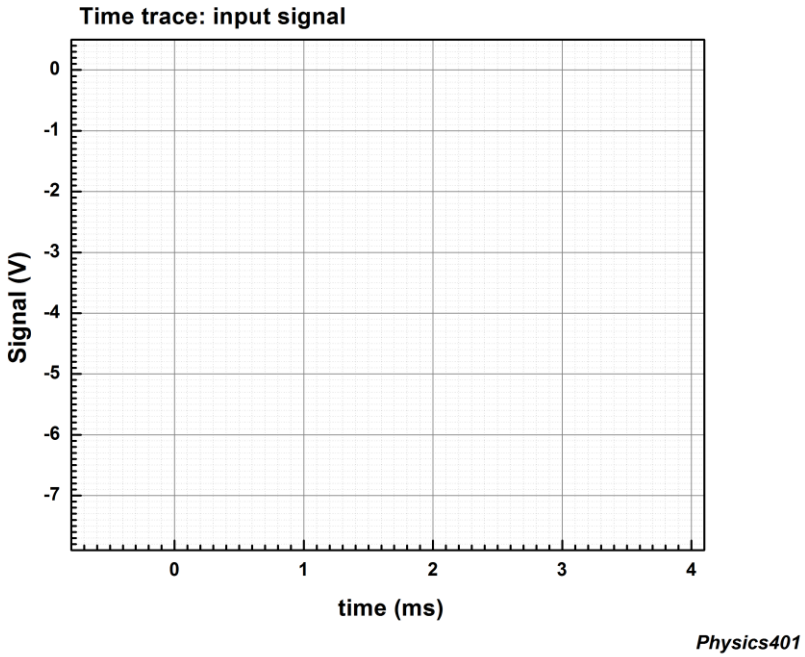
pulse



Appendix #1

Origin templates for the this week Lab:





References:

1. John H. Scofield, “A Frequency-Domain Description of a Lock-in Amplifier” *American Journal of Physics* 62 (2) 129-133 (Feb. 1994).
2. *Steve Smith* “The Scientist and Engineer's Guide to Digital Signal Processing*” copyright ©1997-1998 by Steven W. Smith. For more information visit the book's website at: www.DSPguide.com" **

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